OSCILLATIONS OF A QUASILINEAR NON-AUTONOMOUS SYSTEM WITH ONE DEGREE OF FREEDOM NEAR RESONANCE

(KOLEBANIIA KVAZILINEINYKH NEAVTONOMNYKH SISTEM S ODNOI STEPEN'IU SVOBODY VBLIZI RESONANSA)

PMM Vol.23, No.5, 1959, pp. 851-861

A.P. PROSKURIAKOV
(Moscow)

(Received 30 April 1959)

In an earlier paper [1], there was developed a method for the construction of periodic solutions of a non-autonomous system with one degree of freedom for the case of simple roots of the equations of fundamental amplitudes. In the present work there is considered the general case when the roots of these equations may be multiple roots. A solution containing secular terms is constructed for the case when resonance with unlimited amplitude of oscillations occurs.

1. We shall consider a non-autonomous oscillatory system with one degree of freedom

$$\frac{d^2x}{dt^2} + m^2x = f(t) + \mu F\left(t, x, \frac{dx}{dt}, \mu\right) \tag{1.1}$$

Let us assume that the function $f(t)$ is a continuous function, of period 2π in t, and that its Fourier expansion does not contain harmonics of the mth order (m - an integer). The function $F(t, x, x^{\cdot}, \mu)$ is assumed to be analytic in the variables x, $x^{\cdot}$, μ, and to be a continuous periodic function of period 2π in t. The quantity μ is a small parameter, which for the sake of definiteness we assume to be positive.

Let us separate from the function $F(t, x, x^{\cdot}, \mu)$ the linear term in x and the harmonics of order m:

$$F(t, x, x^{\cdot}, \mu) = F_1(t, x, x^{\cdot}, \mu) + cx + \nu\cos mt + \lambda\sin mt$$

The coefficients c, ν, and λ are assumed to be constants (independent of μ) such that $c \neq 0$, and $\nu^2 + \lambda^2 \neq 0$. The linear system will thus have the frequency k, where k is not an integer. The "perturbation" of the system

$$m^2 - k^2 = c\mu \tag{1.2}$$

is therefore of the order of magnitude of the small parameter μ.

The equation thus generated

$$\frac{d^2x}{dt^2} + m^2x = f(t) \tag{1.3}$$

has a general solution which can be written in the following convenient form:

$$x_0(t) = \varphi(t) + A_0 \cos mt + \frac{B_0}{m} \sin mt \tag{1.4}$$

The function $\phi(t)$ represents the forced oscillations of the system (1.3) under the external force $f(t)$. The last two terms in the formula (1.4) represent the free oscillations of that system. The generated equation thus has a family of periodic solutions depending on two arbitrary constants A_0 and B_0.

We shall seek the periodic solutions of the fundamental equation (1.1) by the use of the small parameter method. We choose the following initial conditions:

$$x(0) = x_0(0) + \beta_1, \quad x^{\cdot}(0) = x_0^{\cdot}(0) + \beta_2 \tag{1.5}$$

where the quantities β_1 and β_2 are functions of μ, which take on the value zero when $\mu = 0$. The solution of (1.1) will thus be of the form

$$x = x(t, \beta_1, \beta_2, \mu)$$

We shall try to determine the structure of the function $x(t, \beta_1, \beta_2, \mu)$. Let us assume that this function has a series expansion in positive powers of the parameters β_1, β_2 and μ. Let us find those terms of this series which are independent of β_1 and β_2 but do depend on μ. It is easily seen that all these terms vanish, except those that are linear in β_1 and β_2. This is due to the fact that the coefficients of these terms satisfy second order linear homogeneous differential equations with vanishing initial conditions. After the terms which are linear in β_1 and β_2 have been computed, the solution of (1.1) can be represented in the form

$$x(t, \beta_1, \beta_2, \mu) = \varphi(t) + A_0 \cos mt + \frac{B_0}{m} \sin mt + \beta_1 \cos mt + \frac{\beta_2}{m} \sin mt +$$
$$+ \sum_{n=1}^{\infty} \left[C_n(t) + \frac{\partial C_n}{\partial \beta_1}\beta_1 + \frac{\partial C_n}{\partial \beta_2}\beta_2 + \frac{1}{2}\frac{\partial^2 C_n}{\partial \beta_1^2}\beta_1^2 + \frac{\partial^2 C_n}{\partial \beta_1 \partial \beta_2}\beta_1\beta_2 + \frac{1}{2}\frac{\partial^2 C_n}{\partial \beta_2^2}\beta_2^2 + \ldots \right]\mu^n \tag{1.6}$$

It is necessary to note that all $C_n(t)$ and their derivatives with respect to β_1 and β_2 are taken when $\beta_1 = \beta_2 = \mu = 0$.

It is not difficult to prove that the following formulas hold for the function $x(t, \beta_1, \beta_2, \mu)$ and its derivatives with respect to time

$$\left(\frac{\partial^{k+l+n} x}{\partial\beta_1^k \partial\beta_2^l \partial\mu^n}\right)_{\beta_1=\beta_2=\mu=0} = \left(\frac{\partial^{k+l+n} x}{\partial A_0^k \partial B_0^l \partial\mu^n}\right)_{\beta_1=\beta_2=\mu=0} \tag{1.7}$$

For $n = 0$ the formulas are obvious. For $n \neq 0$ they can be proved by complete mathematical induction analogous to that corresponding to autonomous systems [2]. For this purpose the following equations are used

$$\left(\frac{\partial^{k+l+n+1} x}{\partial\beta_1^k \partial\beta_2^l \partial\mu^{n+1}}\right)_{\beta_1=\beta_2=\mu=0} = \frac{n+1}{m}\int_0^t \left(\frac{\partial^{k+l+n} F}{\partial\beta_1^k \partial\beta_2^l \partial\mu^n}\right)_{\beta_1=\beta_2=\mu=0} \sin m(t-t_1)\, dt_1$$

These formulas can be obtained by considering the coefficient of $\beta_1^k \beta_2^l \mu^{n+1}$ in the expansion of the function $x(t, \beta_1, \beta_2, \mu)$.

On the basis of the above established property of the function $x(t, \beta_1, \beta_2, \mu)$, one can rewrite the formula (1.6) in the following form:

$$\begin{aligned} x(t, \beta_1, \beta_2, \mu) &= \varphi(t) + A_0 \cos mt + \frac{B_0}{m}\sin mt + \beta_1 \cos mt + \frac{\beta_2}{m}\sin mt + \\ &+ \sum_{n=1}^{\infty}\left[C_n(t) + \frac{\partial C_n}{\partial A_0}\beta_1 + \frac{\partial C_n}{\partial B_0}\beta_2 + \frac{1}{2}\frac{\partial^2 C_n}{\partial A_0^2}\beta_1^2 + \frac{\partial^2 C_n}{\partial A_0 \partial B_0}\beta_1\beta_2 + \frac{1}{2}\frac{\partial^2 C_n}{\partial B_0^2}\beta_2^2 + \ldots\right]\mu^n \end{aligned} \tag{1.8}$$

Hence, for the construction of the function $x(t, \beta_1, \beta_2, \mu)$ one has to know how to compute the coefficient $C_n(t)$ of t^n. The remaining coefficients of the series are then found by successive differentiation of $C_n(t)$ with respect to A_0 and B_0.

The coefficients $C_n(t)$ satisfy the equation

$$\frac{d^2 C_n(t)}{dt^2} + m^2 C_n(t) = H_n(t), \qquad H_n(t) = \frac{1}{(n-1)!}\left[\frac{d^{n-1}F}{d\mu^{n-1}}\right]_{\beta_1=\beta_2=\mu=0}$$

with the initial conditions $C_n(0) = 0$, $C_n^{\cdot}(0) = 0$.

The quantity $dF/d\mu$ is the total partial derivative of the function $F(t, x, x^{\cdot}, \mu)$ with respect to the parameter μ. We obtain

$$C_n(t) = \frac{1}{m}\int_0^t H_n(t_1)\sin m(t-t_1)\, dt_1, \quad C_n^{\cdot}(t) = \int_0^t H_n(t_1)\cos m(t-t_1)\, dt_1 \tag{1.9}$$

In the explicit form, the first three functions $H_n(t)$ are given as follows [2]:

$$H_1(t) = F(t, x_0, x_0^{\bullet}, 0) \qquad (1.10)$$

$$H_2(t) = \left(\frac{\partial F}{\partial x}\right)_0 C_1 + \left(\frac{\partial F}{\partial x^{\bullet}}\right)_0 C_1^{\bullet} + \left(\frac{\partial F}{\partial \mu}\right)_0 \qquad (1.11)$$

$$H_3(t) = \frac{1}{2}\left(\frac{\partial^2 F}{\partial x^2}\right)_0 C_1^2 + \frac{1}{2}\left(\frac{\partial^2 F}{\partial x^{\bullet 2}}\right)_0 C_1^{\bullet 2} + \frac{1}{2}\left(\frac{\partial^2 F}{\partial \mu^2}\right)_0 + \qquad (1.12)$$

$$+ \left(\frac{\partial^2 F}{\partial x \partial x^{\bullet}}\right)_0 C_1 C_1^{\bullet} + \left(\frac{\partial^2 F}{\partial x \partial \mu}\right)_0 C_1 + \left(\frac{\partial^2 F}{\partial x^{\bullet} \partial \mu}\right)_0 C_1^{\bullet} + \left(\frac{\partial F}{\partial x}\right)_0 C_2 + \left(\frac{\partial F}{\partial x^{\bullet}}\right)_0 C_2^{\bullet}$$

The subscript 0 at the parentheses indicates that the symbols x, $x^{\bullet}$, and μ have been replaced by x_0, $x_0^{\bullet}$ and 0, respectively, in the derivatives of the function F.

2. The conditions of periodicity of the function $x(t, \beta_1, \beta_2, \mu)$ and its derivative with respect to time can be expressed in the following form with the aid of the initial conditions (1.5):

$$x(2\pi, \beta_1, \beta_2, \mu) = \varphi(0) + A_0 + \beta_1, \qquad x^{\bullet}(2\pi, \beta_1, \beta_2, \mu) = \varphi^{\bullet}(0) + B_0 + \beta_2 \quad (2.1)$$

Let us substitute $x(2\pi)$ and $x^{\bullet}(2\pi)$ into the left-hand sides of these equations by means of the formula (1.8). After some cancellations there result the equation

$$\sum_{n=1}^{\infty}\Big[C_n(2\pi) + \frac{\partial C_n}{\partial A_0}\beta_1 + \frac{\partial C_n}{\partial B_0}\beta_2 + \frac{1}{2}\frac{\partial^2 C_n}{\partial A_0^2}\beta_1^2 + \frac{\partial^2 C_n}{\partial A_0 \partial B_0}\beta\beta_2 + \frac{1}{2}\frac{\partial^2 C_n}{\partial B_0^2}\beta_2^2 + \ldots\Big]\mu^n = 0 \qquad (2.2)$$

and an analogous one

$$\sum_{n=1}^{\infty}\Big[C_n^{\bullet}(2\pi) + \frac{\partial C_n^{\bullet}}{\partial A_0}\beta_1 + \frac{\partial C_n^{\bullet}}{\partial B_0}\beta_2 + \frac{1}{2}\frac{\partial^2 C_n^{\bullet}}{\partial A_0^2}\beta_1^2 + \frac{\partial^2 C_n^{\bullet}}{\partial A_0 \partial B_0}\beta_1\beta_2 + \frac{1}{2}\frac{\partial^2 C_n^{\bullet}}{\partial B_0^2}\beta_2^2 + \ldots\Big]\mu^n = 0 \qquad (2.3)$$

The functions C_n and $C_n^{\bullet}$ and their derivatives with respect to A_0 and B_0 are taken with $t = 2\pi$, $\beta_1 = \beta_2 = \mu = 0$ in formulas (2.2) and (2.3).

Let us assume that the quantities β_1 and β_2 can be expanded in power series of μ, i.e.,

$$\beta_1 = \sum_{n=1}^{\infty} A_n \mu^n, \qquad \beta_2 = \sum_{n=1}^{\infty} B_n \mu^n$$

We now substitute the expressions for β_1 and β_2 into the left-hand sides of equations (2.2) and (2.3) and express them as power series in μ. Next, we equate to zero the coefficients of these series and arrange the resulting equations in pairs. The terms which are independent of μ yield the following equations

$$C_1(2\pi) = 0, \qquad C_1^{\cdot}(2\pi) = 0 \tag{2.4}$$

The coefficients of the first powers of μ yield the equations

$$C_2(2\pi) + \frac{\partial C_1}{\partial A_0} A_1 + \frac{\partial C_1}{\partial B_0} B_1 = 0, \qquad C_2^{\cdot}(2\pi) + \frac{\partial C_1^{\cdot}}{\partial A_0} A_1 + \frac{\partial C_1^{\cdot}}{\partial B_0} B_1 = 0 \tag{2.5}$$

The coefficients of μ^2 lead to the equations

$$C_3(2\pi) + \frac{\partial C_2}{\partial A_0} A_1 + \frac{\partial C_2}{\partial B_0} B_1 + \frac{\partial C_1}{\partial A_0} A_2 + \frac{\partial C_1}{\partial B_0} B_2 + \frac{1}{2}\frac{\partial^2 C_1}{\partial A_0^2} A_1^2 + \frac{\partial^2 C_1}{\partial A_0 \partial B_0} A_1 B_1 + \frac{1}{2}\frac{\partial^2 C_1}{\partial B_0^2} B_1^2 = 0 \tag{2.6}$$

$$C_3^{\cdot}(2\pi) + \frac{\partial C_2^{\cdot}}{\partial A_0} A_1 + \frac{\partial C_2^{\cdot}}{\partial B_0} B_1 + \frac{\partial C_1^{\cdot}}{\partial A_0} A_2 + \frac{\partial C_1^{\cdot}}{\partial B_0} B_2 + \frac{1}{2}\frac{\partial^2 C_1^{\cdot}}{\partial A_0^2} A_1^2 + \frac{\partial^2 C_1^{\cdot}}{\partial A_0 \partial B_0} A_1 B_1 + \frac{1}{2}\frac{\partial^2 C_1^{\cdot}}{\partial B_0^2} B_1^2 = 0 \tag{2.7}$$

The coefficients of μ^3 yield

$$\begin{aligned} C_4(2\pi) &+ \frac{\partial C_3}{\partial A_0} A_1 + \frac{\partial C_3}{\partial B_0} B_1 + \frac{\partial C_2}{\partial A_0} A_2 + \frac{\partial C_2}{\partial B_0} B_2 + \\ &+ \frac{1}{2}\frac{\partial^2 C_2}{\partial A_0^2} A_1^2 + \frac{\partial^2 C_2}{\partial A_0 \partial B_0} A_1 B_1 + \frac{1}{2}\frac{\partial^2 C_2}{\partial B_0^2} B_1^2 + \frac{\partial C_1}{\partial A_0} A_3 + \frac{\partial C_1}{\partial B_0} B_3 + \\ &+ \frac{\partial^2 C_1}{\partial A_0^2} A_1 A_2 + \frac{\partial^2 C_1}{\partial A_0 \partial B_0} (A_1 B_2 + A_2 B_1) + \frac{\partial^2 C_1}{\partial B_0^2} B_1 B_2 + \\ &+ \frac{1}{6}\frac{\partial^3 C_1}{\partial A_0^3} A_1^3 + \frac{1}{2}\frac{\partial^3 C_1}{\partial A_0^2 \partial B_0} A_1^2 B_1 + \frac{1}{2}\frac{\partial^3 C_1}{\partial A_0 \partial B_0^2} A_1 B_1^2 + \frac{1}{6}\frac{\partial^3 C_1}{\partial B_0^3} B_1^3 = 0 \end{aligned} \tag{2.8}$$

and analogous equations in which the C_n are replaced everywhere by $C_n^{\cdot}$. The other equations can also be written down quite easily.

The equations (2.4) represent the equations for the determination of the constants A_0 and B_0. If these equations have simple roots, the functional determinant

$$\Delta_1 = \begin{vmatrix} \partial C_1 / \partial A_0 & \partial C_1 / \partial B_0 \\ \partial C_1^{\cdot} / \partial A_0 & \partial C_1^{\cdot} / \partial B_0 \end{vmatrix} \tag{2.9}$$

will be different from zero. In this case it is possible to determine A_1 and B_1 from the equations (2.5). Furthermore, by means of equations (2.6) and (2.7), one can find A_2 and B_2, and so on. All these equations are linear in A_n and B_n, and have the same determinant Δ_1.

3. If the equations (2.4) have multiple roots, then

$$\Delta_1 = 0 \tag{3.1}$$

If there is to exist a periodic solution with a finite amplitude in this case, then the following additional condition has to be satisfied:

$$\frac{\partial C_1 / \partial A_0}{\partial C_1^{\cdot} / \partial A_0} = \frac{\partial C_1 / \partial B_0}{\partial C_1^{\cdot} / \partial B_0} = \frac{C_2}{C_2^{\cdot}} \tag{3.2}$$

Making use of condition (3.1), one can eliminate A_2 and B_2 from equations (2.6) and (2.7). Solving the resulting equations simultaneously with (2.5), one can determine A_1 and B_1.

For example, the coefficient A_1 is found to have to satisfy the quadratic equation

$$P_0 A_1^2 + P_1 A_1 + P_2 = 0 \tag{3.3}$$

The coefficients of this equation have the following values:

$$P_0 = \frac{\partial C_1}{\partial B_0}\left[\frac{1}{2}\frac{\partial^2 C_1}{\partial A_0^2}\left(\frac{\partial C_1^{\cdot}}{\partial B_0}\right)^2 - \frac{\partial^2 C_1}{\partial A_0 \partial B_0}\frac{\partial C_1^{\cdot}}{\partial A_0}\frac{\partial C_1^{\cdot}}{\partial B_0} + \frac{1}{2}\frac{\partial^2 C_1}{\partial B_0^2}\left(\frac{\partial C_1^{\cdot}}{\partial A_0}\right)^2\right] - \\ - \frac{\partial C_1^{\cdot}}{\partial B_0}\left[\frac{1}{2}\frac{\partial^2 C_1^{\cdot}}{\partial A_0^2}\left(\frac{\partial C_1}{\partial B_0}\right)^2 - \frac{\partial^2 C_1^{\cdot}}{\partial A_0 \partial B_0}\frac{\partial C_1}{\partial A_0}\frac{\partial C_1}{\partial B_0} + \frac{1}{2}\frac{\partial^2 C_1^{\cdot}}{\partial B_0^2}\left(\frac{\partial C_1}{\partial A_0}\right)^2\right]$$

$$P_1 = \frac{\partial C_1}{\partial B_0}\left[C_2^{\cdot}\left(\frac{\partial^2 C_1}{\partial B_0^2}\frac{\partial C_1^{\cdot}}{\partial A_0} - \frac{\partial^2 C_1^{\cdot}}{\partial B_0^2}\frac{\partial C_1}{\partial A_0} - \frac{\partial^2 C_1}{\partial A_0 \partial B_0}\frac{\partial C_1^{\cdot}}{\partial B_0} + \frac{\partial^2 C_1^{\cdot}}{\partial A_0 \partial B_0}\frac{\partial C_1}{\partial B_0}\right) + \right. \\ \left. + \frac{\partial C_1^{\cdot}}{\partial B_0}\left(\frac{\partial C_2}{\partial A_0}\frac{\partial C_1^{\cdot}}{\partial B_0} - \frac{\partial C_2^{\cdot}}{\partial A_0}\frac{\partial C_1}{\partial B_0} - \frac{\partial C_2}{\partial B_0}\frac{\partial C_1^{\cdot}}{\partial A_0} + \frac{\partial C_2^{\cdot}}{\partial B_0}\frac{\partial C_1}{\partial A_0}\right)\right]$$

$$P_2 = \frac{\partial C_1}{\partial B_0}\left[\frac{1}{2}\frac{\partial^2 C_1}{\partial B_0^2}C_2^{\cdot 2} - \frac{\partial C_2}{\partial B_0}\frac{\partial C_1^{\cdot}}{\partial B_0}C_2^{\cdot} - \left(\frac{\partial C_1^{\cdot}}{\partial B_0}\right)^2 C_3\right] - \\ - \frac{\partial C_1^{\cdot}}{\partial B_0}\left[\frac{1}{2}\frac{\partial^2 C_1^{\cdot}}{\partial B_0^2}C_2^2 - \frac{\partial C_2^{\cdot}}{\partial B_0}\frac{\partial C_1}{\partial B_0}C_2 + \left(\frac{\partial C_1}{\partial B_0}\right)^2 C_3^{\cdot}\right]$$

We note that the coefficients of the equation (3.3) can be represented in different equivalent forms. Knowing the value A_1, it is not difficult to find B_1 by means of one of the equations (2.5).

In order to find the coefficients A_2 and B_2 we multiply the equation (2.8) by $C_2^{\cdot}$. The analogous equation, obtained through a replacement of every C_n by $C_n^{\cdot}$, we multiply by C_2. Next we add the two resulting equations. Then we also add equations (2.6) and (2.7). The system of equations thus obtained will be linear in A_2 and B_2. Let us find the determinant of this system. After some simplifications we obtain

$$\Delta_2^{\cdot} = (C_2 + C_2^{\cdot})\,\Delta_2 \tag{3.4}$$

where

$$\Delta_2 = \frac{\partial C_2}{\partial A_0}\frac{\partial C_1^{\cdot}}{\partial B_0} - \frac{\partial C_2^{\cdot}}{\partial A_0}\frac{\partial C_1}{\partial B_0} - \frac{\partial C_2}{\partial B_0}\frac{\partial C_1^{\cdot}}{\partial A_0} + \frac{\partial C_2^{\cdot}}{\partial B_0}\frac{\partial C_1}{\partial A_0} + \tag{3.5}$$

$$+ A_1\left(\frac{\partial^2 C_1}{\partial A_0^2}\frac{\partial C_1^{\cdot}}{\partial B_0} - \frac{\partial^2 C_1^{\cdot}}{\partial A_0^2}\frac{\partial C_1}{\partial B_0} - \frac{\partial^2 C_1}{\partial A_0\partial B_0}\frac{\partial C_1^{\cdot}}{\partial A_0} + \frac{\partial^2 C_1^{\cdot}}{\partial A_0\partial B_0}\frac{\partial C_1}{\partial A_0}\right) +$$

$$+ B_1\left(\frac{\partial^2 C_1}{\partial A_0\partial B_0}\frac{\partial C_1^{\cdot}}{\partial B_0} - \frac{\partial^2 C_1^{\cdot}}{\partial A_0\partial B_0}\frac{\partial C_1}{\partial B_0} - \frac{\partial^2 C_1}{\partial B_0^2}\frac{\partial C_1^{\cdot}}{\partial A_0} + \frac{\partial^2 C_1^{\cdot}}{\partial B_0^2}\frac{\partial C_1}{\partial A_0}\right)$$

It is not difficult to convince oneself that the systems of equations for the determination of A_n and B_n $(n = 3, 4, \ldots)$ will also be linear. The determinant of all these systems will be $\Delta_2{}^*$.

If the equation (3.3) has two real roots, there will exist two periodic solutions corresponding to a pair of double roots of the equation of the fundamental amplitudes. In this case one can speak of the bifurcation of the solution of the generating equation.

The condition for the existence of triple roots of the equations (2.4) of fundamental amplitudes is the vanishing of the functional determinant that is equal to twice the coefficient P_0 in equation (3.3).

In this case one of the roots of equation (3.3) becomes infinite. Hence, one of the solutions of equation (1.1) will be periodic, while the other will be unbounded.

In all cases when there exists a periodic solution of (1.1), this solution can be represented in the form of a power series in μ:

$$x(t) = x_0(t) + \mu x_1(t) + \mu^2 x_2(t) + \ldots \tag{3.6}$$

The generating solution $x_0(t)$ is determined by formula (1.4). The coefficients $x_n(t)$ are computed by means of the formulas

$$x_1(t) = A_1 \cos mt + \frac{B_1}{m}\sin mt + C_1(t) \tag{3.7}$$

$$x_2(t) = A_2 \cos mt + \frac{B_2}{m}\sin mt + A_1\frac{\partial C_1(t)}{\partial A_0} + B_1\frac{\partial C_1(t)}{\partial B_0} + C_2(t) \tag{3.8}$$

$$x_3(t) = A_3 \cos mt + \frac{B_3}{m}\sin mt + A_2\frac{\partial C_1(t)}{\partial A_0} + B_2\frac{\partial C_1(t)}{\partial B_0} + \frac{1}{2}\frac{\partial^2 C_1(t)}{\partial A_0^2}A_1^2 +$$

$$+ \frac{\partial^2 C_1(t)}{\partial A_0\partial B_0}A_1B_1 + \frac{1}{2}\frac{\partial^2 C_1(t)}{\partial B_0^2}B_1^2 + A_1\frac{\partial C_2(t)}{\partial A_0} + B_1\frac{\partial C_2(t)}{\partial B_0} + C_3(t) \tag{3.9}$$

and so forth. The question on the radius of convergence of the series (3.6) is not considered in this article.

4. Let us consider the stability of the periodic solution of equation (1.1) for the case of multiple roots of the equation of fundamental amplitudes. The equation of variations for equation (1.1) is

$$\frac{d^2y}{dt^2} + m^2y - \mu\left(\frac{\partial F}{\partial x^{\cdot}}\right)_0 \frac{dy}{dt} - \mu\left(\frac{\partial F}{\partial x}\right)_0 y = 0 \qquad (4.1)$$

We denote by $y_1(t)$ and $y_2(t)$ the particular solutions of the equation of variations which form a fundamental system. These solutions satisfy the initial conditions

$$y_1(0) = 1, \qquad y_1^{\cdot}(0) = 0, \qquad y_2(0) = 0, \qquad y_2^{\cdot}(0) = 1$$

Let us consider the characteristic equation for the equation of variations

$$\rho^2 - 2A^*\rho + B^* = 0$$

The coefficients of this equation, as is well known, have the following values

$$A^* = \frac{1}{2}[y_1(2\pi) + y_2^{\cdot}(2\pi)], \qquad B^* = y_1(2\pi)\, y_2^{\cdot}(2\pi) - y_2(2\pi)\, y_1^{\cdot}(2\pi)$$

In order that the periodic solution of equation (1.1) be asymptotically stable, it is necessary and sufficient that the inequality $|\rho| < 1$ be satisfied. For the equation of variations (4.1) this condition reduces to the following two conditions [1]:

$$B^* - 2A^* + 1 > 0. \qquad |B^*| < 1 \qquad (4.2)$$

We shall seek $y_1(t)$ and $y_2(t)$ in the form

$$y_1(t) = y_{10}(t) + \mu y_{11}(t) + \mu^2 y_{12}(t) + \ldots$$
$$y_2(t) = y_{20}(t) + \mu y_{21}(t) + \mu^2 y_{22}(t) + \ldots$$

For the function $y_{10}(t)$, $y_{11}(t)$, $y_{12}(t)$ we have the following equations

$$\frac{d^2y_{10}}{dt^2} + m^2y_{10} = 0, \quad \frac{d^2y_{11}}{dt^2} + m^2y_{11} = \left(\frac{\partial F}{\partial x}\right)_0 y_{10} + \left(\frac{\partial F}{\partial x^{\cdot}}\right)_0 y_{10}^{\cdot}$$

$$\frac{d^2y_{12}}{dt^2} + m^2y_{12} = \frac{1}{2}\left(\frac{\partial^2F}{\partial x^2}\right)_0 y_{10}^2 + \left(\frac{\partial^2F}{\partial x\partial x^{\cdot}}\right)_0 y_{10}y_{10}^{\cdot} + $$
$$+ \frac{1}{2}\left(\frac{\partial^2F}{\partial x^{\cdot 2}}\right)_0 y_{10}^{\cdot 2} + \left(\frac{\partial F}{\partial x}\right)_0 y_{11} + \left(\frac{\partial F}{\partial x^{\cdot}}\right)_0 y_{11}^{\cdot}$$

Analogous equations hold for $y_{20}(t)$, $y_{21}(t)$, $y_{22}(t)$. The initial conditions for all these equations are

$$y_{10}(0) = 1, \quad y_{10}^{\cdot}(0) = 0, \quad y_{1n}(0) = 0, \quad y_{1n}^{\cdot}(0) = 0$$
$$y_{20}(0) = 0, \quad y_{20}^{\cdot}(0) = 1, \quad y_{2n}(0) = 0, \quad y_{2n}^{\cdot}(0) = 0 \qquad (n = 1, 2, 3 \ldots)$$

Solving these equations, we obtain

$$
\begin{aligned}
&y_{10}(t) = \cos mt, \qquad y_{11}(t) = \frac{\partial C_1(t)}{\partial A_0} \\
&y_{12}(t) = \frac{\partial C_2(t)}{\partial A_0} + A_1 \frac{\partial^2 C_1(t)}{\partial A_0^2} + B_1 \frac{\partial^2 C_1(t)}{\partial A_0 \partial B_0} \\
&y_{20}(t) = \frac{1}{m}\sin mt, \quad y_{21}(t) = \frac{\partial C_1(t)}{\partial B_0} \\
&y_{22}(t) = \frac{\partial C_2(t)}{\partial B_0} + A_1 \frac{\partial^2 C_1(t)}{\partial A_0 \partial B_0} + B_1 \frac{\partial^2 C_1(t)}{\partial B_0^2}
\end{aligned} \tag{4.3}
$$

After some computations, the left-hand side of the first of the inequalities (4.2) can be shown to take the form

$$
\begin{aligned}
&B^* - 2A^* + 1 = [y_{11}(2\pi)\, y_{21}^{\cdot}(2\pi) - y_{11}^{\cdot}(2\pi)\, y_{21}(2\pi)]\,\mu^2 + \\
&+ [y_{11}(2\pi) y_{22}^{\cdot}(2\pi) - y_{11}^{\cdot}(2\pi)\, y_{22}(2\pi) + y_{12}(2\pi)\, y_{21}^{\cdot}(2\pi) - \\
&- y_{12}^{\cdot}(2\pi)\, y_{21}(2\pi)]\,\mu^3 + \ldots = \Delta_1\mu^2 + \Delta_2\mu^3 + \ldots
\end{aligned} \tag{4.4}
$$

The quantities Δ_1 and Δ_2 are determined by means of formulas (2.9) and (3.5). In the case of simple roots of the equation of the fundamental amplitudes, one of the conditions for asymptotic stability will be

$$\Delta_1 > 0 \tag{4.5}$$

In case of double roots, this condition is replaced by

$$\Delta_2 > 0 \tag{4.6}$$

In each of these cases, it is necessary to add the second condition of (4.2), which reduces to the inequality

$$\int_0^{2\pi} \left(\frac{\partial F}{\partial x^{\cdot}}\right)_0 dt < 0 \tag{4.7}$$

5. We shall next consider some periodic solutions of equation (1.1). If the parameters A_0 and B_0 are not roots of the equation (2.4), the function $x_n(t)$, which enters into the expansion (3.6), will in general have the form

$$
\begin{aligned}
&x_1(t) = x_1^{(0)}(t) + t x_1^{(1)}(t) \\
&x_2(t) = x_2^{(0)}(t) + t x_2^{(1)}(t) + t^2 x_2^{(2)}(t) \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
&x_n(t) = x_n^{(0)}(t) + t x_n^{(1)} + \ldots + t^n x_n^{(n)}(t)
\end{aligned}
$$

where all the $x_n^{(k)}(t)$ are periodic functions of period 2π in t. Hence, the solution of equation (1.1) in this case will have the following structure:

$$x(t) = \Phi_0^{(1)}(t, \mu) + \mu t\, \Phi_1^{(1)}(t, \mu) + \mu^2 t^2 \Phi_2^{(1)}(t, \mu) + \ldots \tag{5.1}$$

The functions $\Phi_n^{(1)}(t, \mu)$ in formula (5.1) represent periodic functions of period 2π in t, which in the general case do not vanish when $\mu = 0$. All the coefficients A_n and B_n that appear in the initial conditions can be given in advance in this case.

To each simple real root of the equation of fundamental amplitudes there corresponds a unique periodic solution of equation (1.1). If, however, the roots are multiple ones, but the conditions (3.2) are not satisfied, then the equations (2.5) will yield infinite values for the coefficients A_1 and B_1. In this case, there will exist no periodic solution of equation (1.1). We shall now try to find a solution which contains secular terms.

The secular terms cannot occur in the coefficient of the first power of μ in the expansion (3.6), for the equations of fundamental amplitudes are obtained from the condition for the periodicity of this coefficient. Therefore, the secular terms can first appear in the coefficient of μ^2.

The functions $C_n(t)$, which occur in the coefficients $x_n(t)$ of the periodic solution (3.6) of equation (1.1), are periodic functions. This is due to the fact that the quantities $C_n(2\pi)$ and $C_n^{\bullet}(2\pi)$ and their derivatives with respect to A_0 and B_0 are subjected to special conditions. If these conditions are not imposed, then (as is easily verified) the functions $C_n(t)$, determined by formulas (1.9), can be represented in the form

$$C_n(t) = C_n^0(t) + \frac{t}{2\pi}\left[C_n(2\pi)\cos mt + \frac{C_n^{\bullet}(2\pi)}{m}\sin mt\right] \qquad (5.2)$$

where $C_n^0(t)$ is the periodic part of the function.

In the case under consideration, the function $x_2(t)$, will have, in view of (3.8) and (5.2), the following form (the subscript zero has been dropped at $C_1(t)$ and $C_2(t)$):

$$x_2(t) = A_2\cos mt + \frac{B_2}{m}\sin mt + A_1\frac{\partial C_1(t)}{\partial A_0} + B_1\frac{\partial C_1(t)}{\partial B_0} + $$
$$+ C_2(t) + \frac{t}{2\pi}\left(M\cos mt + \frac{N}{m}\sin mt\right) \qquad (5.3)$$

Here

$$M = A_1\frac{\partial C_1}{\partial A_0} + B_1\frac{\partial C_1}{\partial B_0} + C_2, \qquad N = A_1\frac{\partial C_1^{\bullet}}{\partial A_0} + B_1\frac{\partial C_1^{\bullet}}{\partial B_0} + C_2^{\bullet} \qquad (5.4)$$

The coefficients A_n and B_n, beginning with A_1 and B_1, can be given in advance. However, the coefficients A_1 and B_1 are obtainable if one imposes auxiliary conditions on $x_3(t)$. Let us consider the equation for the function $x_3(t)$. Denoting the right-hand side of this equation by $G_3(t)$, we separate the periodic terms which enter into it

$$C_3(t) = \left(\frac{\partial F}{\partial x}\right)_0 x_2 + \left(\frac{\partial F}{\partial x^\cdot}\right)_0 x_2^\cdot + \ldots = \frac{t}{2\pi}\left[M\left(\frac{\partial F}{\partial A_0}\right)_0 + N\left(\frac{\partial F}{\partial B_0}\right)_0\right] + \ldots$$

Let us impose the condition that the function $x_3(t)$ shall not contain secular terms with t^2. This leads to the conditions

$$\int_0^{2\pi}\left[M\left(\frac{\partial F}{\partial A_0}\right)_0 + N\left(\frac{\partial F}{\partial B_0}\right)_0\right]\sin mt dt = 0,$$

$$\int_0^{2\pi}\left[M\left(\frac{\partial F}{\partial A_0}\right)_0 + N\left(\frac{\partial F}{\partial B_0}\right)_0\right]\cos mt\, dt = 0$$

These conditions are equivalent to two equations which determine the coefficients A_1 and B_1:

$$M\frac{\partial C_1}{\partial A_0} + N\frac{\partial C_1}{\partial B_0} + C_2 = 0, \qquad M\frac{\partial C_1^\cdot}{\partial A_0} + N\frac{\partial C_1^\cdot}{\partial B_0} + C_2^\cdot = 0 \tag{5.5}$$

We introduce the following notation:

$$S = \frac{\partial C_1}{\partial A_0} + \frac{\partial C_1^\cdot}{\partial B_0} = \int_0^{2\pi}\left(\frac{\partial F}{\partial x^\cdot}\right)_0 dt \tag{5.6}$$

The solutions of equations (5.5) are

$$A_1 = -\frac{C_2}{S}, \qquad B_1 = -\frac{C_2^\cdot}{S} \tag{5.7}$$

The quantities M and N are given by

$$M = \frac{1}{S}\left(\frac{\partial C_1^\cdot}{\partial B_0}C_2 - \frac{\partial C_1}{\partial B_0}C_2^\cdot\right), \qquad N = \frac{1}{S}\left(-\frac{\partial C_1^\cdot}{\partial A_0}C_2 + \frac{\partial C_1}{\partial A_0}C_2^\cdot\right) \tag{5.8}$$

It is interesting to note that the same result is obtained if one does not impose on the function $x_3(t)$ any conditions, but instead restricts the function $x_{n+1}(t)$ to the same terms which are contained in the preceding function $x_n(t)$, i.e. to the terms with t^{n-1}.

The coefficients A_2 and B_2, and the succeeding ones, cannot be determined from any conditions imposed on the functions $x_n(t)$, for such conditions lead to unbounded values of A_2 and A_2. These coefficients can only be given in advance.

The formulas (5.7) and (5.8) have a meaning only under the condition that the quantity S in (5.6) be different from zero. Thus, the indicated form of the solution is not applicable, in particular, for the case of a conservative system.

Hence, in the considered case, the solution of equation (1.1) has the following structure:

$$x(t) = \Phi_0^{(2)}(t, \mu) + \mu^2 t \Phi_1^{(2)}(t, \mu) + \mu^3 t^2 \Phi_2^{(2)}(t, \mu) + \ldots \tag{5.9}$$

where the functions $\Phi_n^{(2)}(t, \mu)$ are determined in a manner analogous to the one used for the determination of $\Phi_n^{(1)}(t, \mu)$ in formula (5.1).

If conditions (3.2) are fulfilled, then the quantities M and N will vanish and the function $x_2(t)$ will be periodic. The coefficients A_1 and B_1 determined by the formulas (5.7) satisfy equations (2.5). They do not, however, satisfy the infinite system of equations which determine the set of coefficients A_n and B_n. As is shown above, under the condition (3.2) one needs equation (3.3) and one of the equations (2.5) for the determination of the coefficients A_1 and B_1. Should the equations (2.4) have triple roots, one of the solutions of equation (1.1) becomes unbounded. In this case the secular terms can not occur earlier than in the function $x_3(t)$.

The considered cases, when the coefficients A_0 and B_0 satisfy equations (2.4) but the solution of equation (1.1) contains secular terms, are the resonance cases. In addition to those considered, one can point out also other types of resonance, when, for example, the coefficients A_1 and B_1 have finite values, while the coefficients A_2 and B_2 become unbounded, and so forth. This will occur under condition (3.2) if the equations (2.4) have multiple roots and the determinant Δ_2^* becomes zero.

From the above it follows that the basic difference between the resonance solutions and the periodic solutions is the appearance in the resonance solutions of secular terms within the coefficients of μ^2 and of the higher powers of the expansion (3.6), while in the non-resonance, non-periodic solutions these secular terms already appear in the coefficient of the first power of μ.

6. Let us consider some examples*. We make the preliminary remark that all the results presented above also remain valid for the nth order resonance.

1. Oscillations in the neighborhood of the resonance in a regenerative receiver. In this case the equation of oscillations can be reduced to the form

$$\frac{d^2x}{dt^2} + x = \mu\left[\nu\cos t + \lambda\sin t + cx + (\alpha + \beta x + \gamma x^2)\frac{dx}{dt}\right] \tag{6.1}$$

We have the following equations of fundamental amplitudes

* All examples are taken from the book by Malkin [1].

$$\nu + cA_0 + \alpha B_0 + \frac{1}{4}\gamma B_0 (A_0{}^2 + B_0{}^2) = 0$$
$$\lambda + cB_0 - \alpha A_0 - \frac{1}{4}\gamma A_0 (A_0{}^2 + B_0{}^2) = 0 \qquad (6.2)$$

The condition for double roots of equation (6.2) leads to the following relation between the coefficients:

$$27\gamma^2 (\nu^2 + \lambda^2)^2 + 16\alpha\gamma (\alpha^2 + 9c^2)(\nu^2 + \lambda^2) + 64c^2 (\alpha^2 + c^2)^2 = 0 \qquad (6.3)$$

The roots of the equation (6.2) are thus found to be

$$A_0 = \frac{9\gamma\lambda (\nu^2+\lambda^2) + 32\alpha c^2\lambda - 8c(3c^2 - \alpha^2)\nu}{6\alpha\gamma (\nu^2 + \lambda^2) + 16c^2(\alpha^2 + c^2)}$$
$$B_0 = -\frac{9\gamma\nu (\nu^2 + \lambda^2) + 32\alpha c^2\nu + 8c(3c^2 - \alpha^2)\lambda}{6\alpha\gamma (\nu^2 + \lambda^2) + 16c^2(\alpha^2 + c^2)}$$

In the presence of relation (6.3) there exists a resonance solution with secular terms. Periodic solutions will not exist.

2. *Resonance of the second type in a regenerative receiver.* We take the equation of oscillations in the following form:

$$\frac{d^2x}{dt^2} + x = -3\nu\cos 2t - 3\lambda\sin 2t + \mu\left[cx + (\alpha + \beta x + \gamma x^2)\frac{dx}{dt}\right] \qquad (6.4)$$

The amplitude equations will be

$$cA_0 + \alpha B_0 + \frac{1}{2}\beta(\lambda A_0 - \nu B_0) + \frac{1}{2}\gamma B_0\left[\nu^2 + \lambda^2 + \frac{1}{2}(A_0{}^2 + B_0{}^2)\right] = 0$$
$$cB_0 - \alpha A_0 - \frac{1}{2}\beta(\nu A_0 + \lambda B_0) - \frac{1}{2}\gamma A_0\left[\nu^2 + \lambda^2 + \frac{1}{2}(A_0{}^2 + B_0{}^2)\right] = 0 \qquad (6.5)$$

The condition under which there will exist double roots for these equations is

$$\beta^2(\nu^2 + \lambda^2) - 4c^2 = 0 \qquad (6.6)$$

Let us consider some particular cases (the coefficients α and γ have different signs, $A_0{}^2 > 0$, $B_0{}^2 > 0$).

(a) $\nu = 0$. Two sub-cases can arise:

$$A_0 = 0, \qquad B_0{}^2 = -2\lambda^2 - 4\frac{\alpha}{\gamma}, \qquad \beta\lambda = 2c$$
$$B_0 = 0, \qquad A_0{}^2 = -2\lambda^2 - 4\frac{\alpha}{\gamma}, \qquad \beta\lambda = -2c$$

Under these conditions there will exist resonance solutions of the type (5.9). In the presence of the following auxiliary relation

$$5\beta^2(32\alpha - \gamma\lambda^2) = \gamma(60\alpha^2 + 26\alpha\gamma\lambda^2 - 7\gamma^2\lambda^4)$$

between the coefficients, the condition (3.2) will be fulfilled, and, hence, there will exist a periodic solution.

(b) $\lambda = 0$. Here also two sub-cases can occur:

$$A_0 = B_0, \qquad B_0^2 = -\nu^2 - 2\frac{\alpha}{\gamma}, \qquad \beta\nu = 2c$$

$$A_0 = -B_0, \qquad B_0^2 = -\nu^2 - 2\frac{\alpha}{\gamma}, \qquad \beta\nu = -2c$$

This corresponds to the resonance solutions with secular terms. When the additional condition

$$5\beta^2(32\alpha - \gamma\nu^2) = \gamma(60\alpha^2 + 26\alpha\gamma\nu^2 - 7\gamma^2\nu^4)$$

is fulfilled, there will exist a periodic solution.

3. *Duffin's problem in quasilinear formulation.* The equation of oscillations is

$$\frac{d^2x}{dt^2} + x = \mu(\nu\cos t + \lambda\sin t + cx + \gamma x^3) \tag{6.7}$$

The equations of fundamental amplitudes are

$$\nu + cA_0 + \frac{3}{4}\gamma A_0(A_0^2 + B_0^2) = 0, \qquad \lambda + cB_0 + \frac{3}{4}\gamma B_0(A_0^2 + B_0^2) = 0 \tag{6.8}$$

The condition for multiple roots reduces to the following relation between the coefficients of the equation:

$$81\gamma(\nu^2 + \lambda^2) + 16c^3 = 0 \tag{6.9}$$

The coefficients c and γ must have opposite signs. The the roots of the equation (6.8) will be

$$A_0 = -\frac{3}{2}\frac{\nu}{c}, \qquad B_0 = -\frac{3}{2}\frac{\lambda}{c}$$

Under condition (6.9) there can exist no periodic solution.

The examples considered above show that the phenomenon of resonance occurs, as a rule, when $c \neq 0$, i.e. when the "perturbance" of the system (1.2) is not zero. In case of a fundamental resonance this means that the frequency of the natural (characteristic) oscillations of the linear system with resonance, does not usually coincide with the frequency of the disturbing force. In case of nth type of resonance the frequency of the natural (characteristic) oscillations with resonance, is usually not $1/n$ times the frequency of the disturbing force (n being an integer).

BIBLIOGRAPHY

1. Malkin, I.G., *Nekotorye zadachi teorie nelineinykh kolebanii (Some Problems in the Theory of Non-linear Oscillations)*. Gostekhizdat, 1956.

2. Proskuriakov, A.P., Postroenie periodicheskikh reshenii avtonomnykh sistem s odnoi stepen'iu svobody v sluchae proizvol'nykh veshchestvennykh kornei uravneniia osnovnykh amplitud (Construction of periodic solutions of autonomous systems with one degree of freedom for the case of arbitrary real roots of the equation of fundamental amplitudes). *PMM* Vol. 22, No. 4, pp. 510-518, 1958.

Translated by H.P.T.